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Gaussian Quadrature Formulae with Fixed Nodes

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Let $N=\sum_{i=1}^{k}(\mu_i+1)+\sum_{i=1}^{k} m_i$ and let $\{u_i\}_{i=1}^{N}$ be an extended Chebysev system. Let $\mu_1, \mu_2, ..., \mu_k$ be given odd positive integers and $m_1, m_2, ..., m_s$ given even positive integers with corresponding fixed nodes $x_1<x_2<\cdots<x_s$ in the interval (a, b). If L is a positive linear functional on $U=\operatorname{span}\{u_i\}_{i=1}^{N}$, we prove that L has a unique representation of the form

$$Q(p)=\sum_{i=1}^{s}\sum_{j=1}^{m_i} a_{ij}\,p\Big[\underbrace{x_1,...,x_1}_{m_1},\underbrace{x_2,...,x_2}_{m_2},...,\underbrace{x_i,...,x_i}_{j}\Big]$$
$$+\sum_{i=1}^{k}\sum_{j=1}^{\mu_i} b_{ij}\,p\Big[\underbrace{x_1,...,x_1}_{m_1},\underbrace{x_2,...,x_2}_{m_2},...,\underbrace{x_s,...,x_s}_{m_s},\underbrace{t_1,...,t_1}_{\mu_1},...,\underbrace{t_i,...,t_i}_{j}\Big]$$

such that $Q(p)=L(p)$ for all $p\in U$. Here $a<t_1<t_2<\cdots<t_k<b$. The proof combines use of the degree of a mapping with results from the theory of divided differences. © 1988 Academic Press, Inc.

I. INTRODUCTION

Given a positive linear functional L on an extended Chebysev system $U=\operatorname{span}\{u_i\}_{i=1}^{N}$, Barrow [3] has shown that there exist unique nodes $\{t_i\}_{i=1}^{k}$ and coefficients $\{a_{ij}\}$ such that the corresponding quadrature formula is exact over the system; i.e., $\sum_{i=1}^{k}\sum_{j=0}^{\mu_i-1} u^{(j)}(t_i)=L(u)$ for all $u\in U$. Here $\{\mu_i\}$ are given odd positive integers and $\dim U=\sum_{i=1}^{k}(\mu_i+1)$. Such a formula is known as a Gaussian quadrature formula and has been shown to be of maximal precision.

This paper concerns a generalization of formulas of this type. One is allowed to initially choose points x in the interval (a, b), and given the preassigned multiplicities of these and the free nodes, a quadrature formula is shown to exist.

In a departure from previous quadrature formulas, the divided difference has been used in place of derivative evaluations. This is primarily due to the fact that it is possible for free knots to combine with fixed nodes.

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However, as has historically been the case, the free knots remain separate from each other as in [1], [2].

The following section provides background in total positivity, divided differences, and topological theory which will be used in the proof of the main theorem. The last section follows the basic outline of Barrow's work [3], proving the main result of the paper. As with the theorem of Barrow, this formula is of maximal precision.

II. Definitions and Preliminary Results

Let $\{p_i\}_{i=1}^N \in C^{N-1}[a, b]$ form an extended Chebysev (ET) system, i.e.,

$$D^* \begin{vmatrix} t_1\, t_2 \cdots t_N \\ p_1\, p_2 \cdots p_N \end{vmatrix} = \det\{p_i(t_j)\}_{i,j=1}^N > 0 \qquad \text{whenever } a \leqslant t_1 \leqslant t_2 \cdots \leqslant t_N \leqslant b. \tag{2.1}$$

Here the "*" means that if some of the t_i's coincide, then the columns in the matrix corresponding to coincident t_i's are replaced by derivatives of increasing order (see Ref. [5] for details).

A linear function L on U is said to be positive if whenever $p \in U$ is nontrivial and nonnegative, $L(p) > 0$. L is called nonnegative if $L(p) \geqslant 0$ whenever $p \geqslant 0$. It is shown in Krein and Rutman [6] that a nonnegative linear functional on U may be extended to a nonnegative linear functional on $C[a, b]$. Hence, by the Riesz representation theorem for the dual of $C[a, b]$ we may assume that $L(p) = \int_a^b p(t)\, d\sigma(t)$ for all $p \in U$, where σ is a nondecreasing right-continuous bounded function.

We now provide a brief discussion of divided differences, referring the interested reader to Schumaker [7] for a more thorough treatment. For the remainder of this paper we use the following definition:

Given points $t_1, ..., t_{r+1}$ and a function f, define the rth-order divided difference of f over these points by

$$f[t_1, ..., t_{r+1}] = \frac{D^* \left|{}^{t_1, ..., t_{r+1}}_{1, x, ..., x^{r-1}, f}\right|}{D^* \left|{}^{t_1, ..., t_{r+1}}_{1, x, ..., x^r}\right|}. \tag{2.2}$$

Here the "*" allows for coincident t's as in definition (2.1).

We use the special notation $V(t_1, ..., t_{r+1})$ for the Vandermonde determinant $D^* \left|{}^{t_1, ..., t_{r+1}}_{1, x, ..., x^r}\right|$. We have not specified the ordering of $\{t_i\}_{i=1}^{r+1}$ as in fact it does not matter. The recursive nature of the divided difference will be used often, i.e., given any points $\{t_i\}_{i=1}^{r+1}$, $t_{r+1} \neq t_1$ (see Ref. [4]),

$$f[t_1, ..., t_{r+1}] = \frac{f[t_2, ..., t_{r+1}] - f[t_1, ..., t_r]}{t_{r+1} - t_1}. \tag{2.3}$$

The continuity of the divided difference allows for the following fact: Let $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{r+1}$. Fix $1 \leqslant i \leqslant r+1$ and suppose

$$t_1, ..., t_{r+1} = \overbrace{\tau_1, ..., \tau_1}^{l_1} < \overbrace{\tau_2, ..., \tau_2}^{l_2} < \cdots < \overbrace{\tau_d, ..., \tau_d}^{l_d}.$$

Then

$$\frac{\partial}{\partial \tau_i} f[t_1, ..., t_{r+1}] = l_i f[\underbrace{\tau_1, ..., \tau_1}_{l_1}, ..., \underbrace{\tau_i, ..., \tau_i}_{l_i+1}, ..., \underbrace{\tau_d, ..., \tau_d}_{l_d} \quad \text{for } i = 1, 2, ..., d. \tag{2.4}$$

The correlation between the divided difference of a function and its derivative values is often exploited. With $\{\tau_i\}_{i=1}^d$ as above,

$$f[t_1, ..., t_{r+1}] = \sum_{i=1}^{d} \sum_{j=1}^{l_i} \alpha_{ij} D^{j-1} f(\tau_i), \tag{2.5}$$

where

$$\alpha_{il_i} = \frac{V(\overbrace{\tau_1, ..., \tau_1}^{l_1}, ..., \overbrace{\tau_i, ..., \tau_i}^{l_i-1}, ..., \overbrace{\tau_d, ..., \tau_d}^{l_d})}{V(t_1, ..., t_{r+1})}.$$

It is worthwhile at this point to note that the main result holds if we replace this definition of the divided difference with the most generalized definition (see Ref. [7, p. 81]).

The following is a discussion of the elements of topological degree theory which will be needed:

Let $D \subseteq R^N$ be a bounded open set and let $F: \bar{D} \to R^N$ be continuous. Then if $c \in R^N$ and $c \notin F(\partial D)$, where ∂D means the boundary of D, then the degree of F with respect to D and c is defined, is an integer, and will be denoted $\deg(F, D, c)$. We list below some of the relevant properties of $\deg(F, D, c)$ [3].

(i) If $F \in C^1(D) \cap C(\bar{D})$, $c \notin F(\partial D)$, and $\det[F'(x)] \neq 0$ when $F(x) = c$, then there are a finite number of points $x_i \in D$ where $F(x_i) = c$. Moreover, $\deg(F, D, c) = \sum_i \operatorname{sgn}[\det[F'(x_i)]]$.

(ii) If $\deg(F, D, c) \neq 0$, there is at least one solution in D to the equation $F(x) = c$.

(iii) If $F: \bar{D} \times [0, 1] \to R^N$ is continuous ad $F(x, \lambda) \neq c$ for $x \in \partial D$, $0 \leqslant \lambda \leqslant 1$, then $\deg[F(\cdot, \lambda), D, c]$ is constant, independent of λ.

We now define the concepts which are necessary for the statement of the main theorem:

Let $\mu_1, \mu_2, ..., \mu_k$ be odd positive integers, $m_1, m_2, ..., m_s$ even positive integers, and let

$$N = \sum_{i=1}^{k} (\mu_i + 1) + \sum_{i=1}^{s} m_i.$$

Let U be an ET-system of order N and let L be a given positive linear functional on $U = \operatorname{span}\{p_i\}_{i=1}^{N}$. When possible we will write

$$p[\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_i, ..., t_i}_{j}] \text{ for } p[\underbrace{x_1, ..., x_1}_{m_1}, ..., \underbrace{x_s, ..., x_s}_{m_s}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_i, ..., t_i}_{j}]$$

to eliminate cumbersome notation.

Let $\Delta_k \in R^k$ be defined by $\Delta_k \equiv \{(t_1, t_2, ..., t_k) = \mathbf{t}: a < t_1 < t_2 < \cdots < t_k < b\}$. Let $\Delta_{k,\varepsilon} \in R^k$ be defined by $\Delta_{k,\varepsilon} = \{\mathbf{t} \in \Delta_k: |t_{i+1} - t_i| > \varepsilon$ for $i = 1, 2, ..., k-1\}$. Finally, let $(\mathbf{a}, \mathbf{b}) \equiv (a_{1_1}, a_{1_2}, ..., a_{1_{m_1}}, a_{2_1}, a_{2_2}, ..., a_{2_{m_2}}, ..., a_{s_{m_s}}, b_{1_1}, b_{1_2}, ..., b_{1_{\mu_1}}, b_{2_1}, ..., b_{2_{\mu_2}}, ..., b_{k_{\mu_k}})$.

III. Gaussian Quadrature with Fixed Nodes

This section contains the proof of the following theorem:

THEOREM 1. *Let $a < x_1 < \cdots x_s < b$, $\{u_i\}_{i=1}^{k}$ and $\{m_i\}_{i=1}^{s}$ be given as above. Then there is a unique $\mathbf{t} \in \Delta_{k,\varepsilon}$ where $\varepsilon > 0$ and there is a unique coefficient vector $(\mathbf{a}, \mathbf{b})$ such that*

$$L(p) = \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_{ij}\, p[\underbrace{x_1, ..., x_1}_{m_1}, ..., \underbrace{x_i, ..., x_i}_{j}]$$

$$+ \sum_{i=1}^{k} \sum_{j=1}^{\mu_i} b_{ij} p\, [\underbrace{x_1, ..., x_1}_{m_1}, ..., \underbrace{x_6, ..., x_s}_{m_s}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_i, ..., t_i}_{j}] \quad (3.1)$$

for all $p \in U$. The proof is based upon a series of lemmas.

Note. By Proposition 1 of Barrow [3] this formula is of maximal precision.

LEMMA 1. *Let $a < x_1 < x_2 < \cdots < x_s < b$ be fixed and let $\mathbf{t} \in \Delta_k$ be given. Then for $i = 1, 2, ..., s+k$ we define the "generalized Lagrange polynomials" $P_{ij} \in U$ as follows: For $i = 1, 2, ..., s$ and $j = 1, ..., m_i$ define $p_{ij} \in U$ by*

$$p_{ij}\, [\underbrace{x_1, ..., x_1}_{m_1}, ..., \underbrace{x_l, ..., x_l}_{n}] = 0 \qquad \textit{for} \quad (i, j) \neq (l, n),$$

where $l = 1, 2, ..., s$, $n = 1, ..., m_l$,

$$p_{ij}\,[\underbrace{x_1, ..., x_1}_{m_1}, ..., \underbrace{x_i, ..., x_i}_{j}] = 1$$

and

$$p_{ij}\,[\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_l, ..., t_l}_{n}] = 0 \qquad for \quad l = 1, 2, ..., k,\ n = 1, ..., \mu_l + 1.$$

For $i = s+1, ..., s+k$ *and* $j = 1, ..., \mu_i + 1$ *define* $p_{ij} \in U$ *by*

$$p_{ij}[\underbrace{x_1, ..., x_1}_{m_1}, ..., \underbrace{x_l, ..., x_l}_{n}] = 0 \qquad for\ l = 1, 2, ..., s,\ n = 1, ..., m_l$$

$$p_{ij}[\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_l, ..., t_l}_{n}] = 0 \qquad for\ (i,j) \neq (l+s, n),\ l = 1, 2, ..., k,\ n = 1, ..., \mu_l + 1$$

and

$$P_{ij}\,[\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_{i-s}, ..., t_{i-s}}_{j}] = 1.$$

Such a polynomial can be seen to exist and be unique by writing out the $N \times N$ *matrix equation which determines it. Let*

$$a_{ij} = L[p_{ij}(\cdot\,; \mathbf{t})] \qquad for \quad i = 1, 2, ..., s,\ j = 1, ..., m_i$$

$$b_{ij} = L[p_{i+s,j}(\cdot\,; \mathbf{t})] \qquad for \quad i = 1, 2, ..., k,\ j = 1, ..., \mu_i.$$

Then $\mathbf{t}$ *and* $(\mathbf{a}, \mathbf{b})$ *satisfy* (3.1) *if and only if*

$$L(p_{i,\mu_{i-s}^{+1}}(\cdot\,; \mathbf{t})) = 0 \qquad for \quad i = s+1, ..., s+k. \tag{3.2}$$

Proof. If (3.1) holds then by definition of $p_{i,\mu_{i-s}^{+1}}(\cdot\,; \mathbf{t})$, the right-hand side of (3.1) is identically zero for $i = s+1, ..., s+k$. Thus (3.2) holds.

Conversely, if (3.2) holds, then by the definition of the remaining N functions $P_{i,j}$, (3.1) holds for all $P_{i,j}$. As we have a basis for U in these functions, (3.1) holds for all $p \in U$.

DEFINITION. Fix $a < r_1 < r_2 < \cdots < r_k < b$ separate from $\mathbf{x}$ and define $L_0(p) = \sum_{i=1}^{k} p(r_i)$. Then for $0 \leqslant \lambda \leqslant 1$ define $L_\lambda(p) = \lambda L_0(p) + (1-\lambda)\, L(p)$.

Remark. It is useful at this point to examine the functions

$p_{i+s,\mu_i+1}(\cdot;\mathbf{t})$ which we redefine as $p_i(\cdot;\mathbf{t})$ for $i=1,2,\dots,k$. By Eqs. (2.3), (2.4), and (2.5),

$$\begin{aligned} p_i^{(j)}(x_l;\mathbf{t})&=0 \qquad \text{for} \quad l=1,\dots,s, j=0,1,\dots,m_l-1\\ p_i^{(j)}(t_l;\mathbf{t})&=0 \qquad \text{for} \quad l\neq i, j=0,1,\dots,\mu_l\\ p_i^{(j)}(t_i;\mathbf{t})&=0 \qquad \text{for} \quad j=0,1,\dots,\mu_i-1 \end{aligned} \tag{3.3}$$

and

$$p_i^{(\mu_i)}(t_i;\mathbf{t})=\frac{V(\mathbf{x},\overbrace{t_1,\dots,t_1}^{\mu_1},\dots,\overbrace{t_i,\dots,t_i}^{\mu\cdot+1})}{V(\mathbf{x},\underbrace{t_1,\dots,t_1}_{\mu_1},\dots,\underbrace{t_i,\dots,t_i}_{\mu\cdot})}>0.$$

LEMMA 2. *Let $a<x_1<x_2<\cdots<x_s<b$, and $\{m_i\}_{i=1}^s$ and $\{\mu_i\}_{i=1}^k$ be as above. If $\mathbf{t}_\lambda$ and $(\mathbf{a}^\lambda,\mathbf{b}^\lambda)$ satisfy* (3.1) *for $0\leqslant\lambda\leqslant 1$ there exists an $\varepsilon>0$ such that $\mathbf{t}_\lambda\in\Delta_{k,\varepsilon}$.*

Proof. Assume to the contrary that no such $\varepsilon>0$ exists. Then there exists a sequence $t^n\subseteq\Delta_k$ such that $\mathbf{t}^n\to\mathbf{t}\in\partial\Delta_k$ as $n\to\infty$. Without loss of generality, we can assume that $t_1^n\to t_2$. Denote the corresponding sequence by $\{\lambda_n\}$, so that as $\lambda_n\to\lambda_0\in[0,1]$, $t_1^n\to t_1$ and $t_2^n\to t_1$.

Construct a sequence of polynominals $p_n\in U$ such that

$$p_n[\underbrace{x_1,\dots,x_1}_{m_1},\dots,\underbrace{x_i,\dots,x_j}_{j}]=0, \qquad j=1,2,\dots,m_i,\ i=1,2,\dots,s$$

$$p_n[\mathbf{x},\underbrace{t_1^n,\dots,t_1^n}_{\mu_1},\underbrace{t_i^n,\dots,t_i^n}_{j}=0, \qquad i=1,2, j=1,2,\dots,\mu_i$$

$$p_n[\mathbf{x},\underbrace{t_1^n,\dots,t_1^n}_{\mu_1},\dots,\underbrace{t_i^n,\dots,t_j^n}_{j}]=0, \qquad i=3,4,\dots,k, j=1,2,\dots,\mu_i+1$$

$$p_n(a)=0, \qquad p_n'(a)>0, \qquad \|p_n\|_\infty=1.$$

Then for each n, p_n has a full set of zeros and is completely determined. Since $\mathbf{t}^n$ and $(\mathbf{a}^{\lambda_n},\mathbf{b}^{\lambda_n})$ give a formula (3.1) for L_{λ_n}, $L_{\lambda_n}(p_n)=0$ for all n.

Define $p\in U$ by

$$p[\underbrace{x_1,\dots,x_1}_{m_1},\dots,\underbrace{x_i,\dots,x_i}_{j}]=0, \qquad j=1,2,\dots,m_i,\ i=1,2,\dots,s$$

$$p[\mathbf{x},\underbrace{t_1,\dots,t_1}_{j}]=0, \qquad j=1,2,\dots,\mu_1+\mu_2$$

$$p[\mathbf{x},\underbrace{t_1,\dots,t_1}_{\mu_1+\mu_2},\dots,\underbrace{t_i,\dots,t_i}_{j}=0, \qquad i=3,4,\dots,k, j=1,2,\dots,\mu_i+1.$$

By continuity and Rolle's theorem, $\lim_{n\to\infty} p_n = p$. By construction the zeros of p are all even and thus $p \geqslant 0$. Notice that the possibility of $t_1 = x_i$ for $i = 1, 2, ..., s$ will nevertheless produce a zero with even multiplicity. Therefore, $p > 0$ and so $L_{\lambda_0}(p) > 0$.

However, by construction of $\{p_n\}$, $L_{\lambda_n}(p_n) = 0$ for all n. As $\lim_n L_{\lambda_n}(p_n) = L_{\lambda_0}(p)$ by the continuity of $L_\lambda(p)$, we have the desired contradiction.

Let $\varepsilon > 0$ be as in Lemma 2. For $\mathbf{t} \in \Delta_{k,\varepsilon}$, let $p_i(\cdot; \mathbf{t}) \in V$ be the polynomial $p_{i_{\mu_i}+1}(\cdot; \mathbf{t})$ of Lemma 1. Define the map $F: \Delta_{k,\varepsilon} \times [0, 1] \to R^k$ by $F_i(\mathbf{t}; \lambda) = -L_\lambda[p_i(\cdot; \mathbf{t})]$ for $i = 1, 2, ..., k$. Then F is continuous in $(\mathbf{t}; \lambda)$ and Lemmas 1 and 2 imply that $F(\mathbf{t}; \lambda) \neq 0$ if $\mathbf{t} \in \partial\Delta_{k,\varepsilon}$.

We need the following fact in order to compute $\partial D/\partial \mathbf{t}$.

LEMMA 3. *For* $1 \leqslant m \leqslant k$, *and* $p_m(\cdot; \mathbf{t}) \in U$ *defined as above,*

$$b_{m\mu_m} V_0^m + \sum_{i=m+1}^{k} \sum_{j=1}^{\mu_i} b_{ij}(-1)^{(\sum_{\alpha=m+1}^{i-1} \mu_\alpha)+j+1} V_{ij}^m > 0, \tag{3.4}$$

where $\mathbf{t}$ *and* $(\mathbf{a}, \mathbf{b})$ *satisfy* $F(\mathbf{t}, \lambda) = 0$ for $0 \lesssim \lambda \lesssim 1$. *We define*

$$V_0^m = \frac{V(\mathbf{x}, \overbrace{t_1, ..., t_1}^{\mu_1}, ..., \overbrace{t_m, ..., t_m}^{\mu_m - 1})}{V(\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_m, ..., t_m}_{\mu_m})}$$

and

$$V_{i,j}^m = \frac{V(\mathbf{x}, \overbrace{t_1, ..., t_1}^{\mu_1}, ..., \overbrace{t_m, ..., t_m}^{\mu_m - 1}, ..., \overbrace{t_i, ..., t_i}^{j})}{V(\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_m, ..., t_m}_{\mu_m}, ..., \underbrace{t_i, ..., t_i}_{j}))}.$$

Proof. For $1 \leqslant l \leqslant k$ construct $\hat{p}_m \in U$ by

$$\begin{aligned}
\hat{p}_m^{(j)}(x_i) &= 0, && i = 1, ..., s, j = 0, 1, ..., m_i - 1\\
\hat{p}_m^{(j)}(t_i) &= 0, && i \neq m, i = 1, ..., k, j = 0, 1, ..., \mu_i\\
\hat{p}_m^{(j)}(t_m) &= 0, && j = 0, 1, ..., \mu_m - 2\\
\hat{p}_m^{(\mu_m - 1)}(t_m) = p_m^{(\mu_m)}(t_m) &> 0 && \text{as defined in Lemma 1, see Eqs. (3.3),}\\
p_m(a) &= 0 && \text{and} \quad \|p_m\|_\infty = 1.
\end{aligned}$$

If $t_i = x_j$ for some i, j, replace μ_i by $m_j + \mu_i$.

Then by construction $\hat{p}$ has $N - 1$ zeros, and as the interior zeros are all even, $\hat{p}_m \geqslant 0$ on $[a, b]$. We will now show that $Q(\hat{p}_m)$ is exactly (3.4), which must be strictly positive by the nature of $\hat{p}_m$.

By construction $\hat{p}_m[\underbrace{x_1, ..., x_1}_{m_1}, ..., \underbrace{x_i, ..., x_i}_{j}] = 0$ for $i = 1, 2, ..., s$ and $j = 1, 2, ..., m_i$. Similarly for $i < m$,

$$\hat{p}_m[\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_i, ..., t_i}_{j}] = \sum_{\beta=1}^{i-1} \sum_{l=0}^{\mu_\beta - 1} \alpha_{\beta l} D^{(l)} \hat{p}_m(t_\beta) + \sum_{l=0}^{j-1} \alpha_{il} D^{(l)} \hat{p}_m(t_i) = 0 \qquad \text{for} \quad j = 1, ..., \mu_i.$$

For the case $i = m$, by (2.2), $\hat{p}_m[\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_m, ..., t_m}_{j}]$ equals

$$\frac{\begin{vmatrix} 1 & 1 \cdots 0 & 1 & 0 & \cdots\cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots 0 \hat{p}_m(t_m) \hat{p}'_m(t_m) \cdots \hat{p}^{(j-1)}(t_m) \end{vmatrix}}{V(\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_m, ..., t_m}_{j})} = 0 \qquad \text{for} \quad j = 1, ..., \mu_m - 1$$

and equals

$$\hat{p}_m^{(\mu_m - 1)}(t_m) \frac{V(\mathbf{x}, \overbrace{t_1, ..., t_1}^{\mu_1}, ..., \overbrace{t_m, ..., t_m}^{\mu_m - 1})}{V(\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_m, ..., t_m}_{\mu_m})} \qquad \text{for} \quad j = \mu_m.$$

For $i > m$,

$$\hat{p}_m[\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_i, ..., t_i}_{j}] = \frac{\begin{vmatrix} 1 & 0 \cdots\cdots & \cdots\cdots & 1 & \cdots\cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots 0 \hat{p}^{(\mu_m - 1)}(t_m) \cdots \hat{p}(t_i) \cdots \hat{p}^{(j-1)}(t_i) \end{vmatrix}}{V(\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_i, ..., t_j}_{j})}$$

$$= \hat{p}^{(\mu_m - 1)}(t_m)(-1)^{(\sum_{\alpha=m+1}^{i-1} \mu_\alpha) + j + 1} V_{ij}^m \qquad \text{for} \quad j = 1, 2, ..., \mu_i.$$

Therefore as $Q(\hat{p}_m) = L(\hat{p}_m) > 0$,

$$Q(p_m) = b_{m\mu_m} \hat{p}^{(\mu_m - 1)}(t_m) V_0^m + \sum_{i=m+1}^{k} \sum_{j=1}^{\mu_i} b_{ij} \hat{p}^{(\mu_m - 1)}(t_m)(-1)^{\sum_{\alpha=m+1}^{i-1} \mu_\alpha + j} V_{ij}^m$$
$$> 0 \qquad \text{for} \quad j = 1, 2, ..., \mu_i.$$

As $\hat{p}_m^{(\mu_m - 1)}(t_m) = p_m^{(\mu_m)}(t_m) > 0$, the result is shown.

LEMMA 4. *If* $\mathbf{t}$ *and* $(\mathbf{a}, \mathbf{b})$ *satisfy* $F(\mathbf{t}, \lambda)=0$ *for* $0<\lambda\leqslant 1$, *then*

$$\det \frac{\partial F}{\partial \mathbf{t}}(\mathbf{t}, \lambda)>0.$$

Proof. As in the work of Barrow [3] we compute the difference quotient, omitting reference to λ. The determinant definition for the divided difference seems to give the greatest insight into this computation.

Let $|h|$ be small and $\mathbf{e}_1=(1, 0, ..., 0)\in R^k$. Then

$$\begin{aligned} F_1(\mathbf{t}+h\mathbf{e}_1)-F_1(\mathbf{t}) &= -L_\lambda(p_1(\mathbf{t}+h\mathbf{e}_1)) \\ &= -\sum_{i=1}^{m_s}\sum_{j=1}^{\mu_i} a_{ij}p_1[\underbrace{x_1, ..., x_1}_{m_1}, ..., \underbrace{x_i, ..., x_i}_{j}; \mathbf{t}+h\mathbf{e}_1] \\ &\quad -\sum_{i=1}^{k}\sum_{j=1}^{\mu_i} b_{ij}p_1[\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_i, ..., t_i}_{j}; \mathbf{t}+h\mathbf{e}_1]. \end{aligned}$$

By the definition of p_1, the first set of sums is zero. We break up the second sum at t_1 to obtain

$$\begin{aligned} F_1(\mathbf{t}+h\mathbf{e}_1)-F_1(\mathbf{t}) &= -\sum_{j=1}^{\mu_1-1} b_{1j}p_1[\mathbf{x}, \underbrace{t_1, ..., t_1}_{j+1}; \mathbf{t}+h\mathbf{e}_1] \\ &\quad -\sum_{i=2}^{k}\sum_{j=0}^{\mu_i-1} b_{ij}p_1[\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_i, ..., t_i}_{j+1}; \mathbf{t}+h\mathbf{e}_1]. \end{aligned}$$

For $j=1, 2, ..., \mu_1$,

$$p_1[\mathbf{x}, \underbrace{t_1, ..., t_1}_{j}; \mathbf{t}+h\mathbf{e}_1]$$

$$= \frac{\begin{vmatrix} 1\,0 & \cdots & 0 & 1 & \cdots\cdots 0 & \cdots & 0 \\ \vdots & & \vdots & & & & \vdots \\ p_1(x_1) & \cdots & p_1^{(m_s-1)}(x_S) & p_1(t_1; \mathbf{t}+h\mathbf{e}_1) & \cdots & & p_1^{(j-1)}(t_1; \mathbf{t}+h\mathbf{e}_1) \end{vmatrix}}{V(\mathbf{x}, \underbrace{t_1, ..., t_1}_{j}))}.$$

Using Taylor's theorem we expand $p_1(x; \mathbf{t}+h\mathbf{e}_1)$ about $\mathbf{t}+h\mathbf{e}_1$. We define

$$\begin{aligned} A_j &= \frac{p_1^{(j-1)}(t_1; \mathbf{t}+h\mathbf{e}_1)}{h} \\ &= p_1^{(\mu_1)}(t_1+h; \mathbf{t}+h\mathbf{e}_1)\frac{(-h)^{\mu_1-j}}{(\mu_1-j+1)!}+o(h^{\mu_1-j}) \end{aligned}$$

so that

$$\lim_{h\to 0} p_1[\mathbf{x}, \underbrace{t_1, ..., t_1}_{j}; \mathbf{t}+h\mathbf{e}_1]$$

$$= \lim_{h\to 0} \frac{\begin{vmatrix} 1\, 0 \cdots & 1 & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 \cdots & 0\, A_1 A_2 & \cdots A_j \end{vmatrix}}{V(\mathbf{x}, \underbrace{t_1, ..., t_1}_{j})}.$$

We note that

$$\lim_{h\to 0} A_j = \begin{cases} 0 & \text{for } j=1, 2, ..., \mu_1 - 1 \\ -p_1^{(\mu_1)}(t_1; \mathbf{t})\, V_0^1 & \text{for } j=\mu_1. \end{cases}$$

For the second set of terms in the sum, we use the fact that $p_1^{(j-1)}(t_i; \mathbf{t}+h\mathbf{e}_1)=0$ for $j=1, 2, ..., \mu_i$ and $i \neq 1$. This leaves the only nonzero element in the last row of the corresponding determinant of $-p^{(\mu_1)}(t_1; \mathbf{t})$. Thus for $i=2, ..., k$ and $j=1, 2, ..., \mu_i$,

$$\lim_{h\to 0} \frac{p_1}{h} [\mathbf{x}, \underbrace{t_1, ..., t_1}_{\mu_1}, ..., \underbrace{t_i, ..., t_j}_{j}; \mathbf{t}+h\mathbf{e}_1]$$

$$= (-1)^{\sum_{l=2}^{i-1} \mu_l + j} (-p_1^{(\mu_1)}(t_1; \mathbf{t}))\, V_{i,j}^1,$$

where the sign is contributed by the placement of $p_1^{(\mu_1)}$ in the last row. Thus

$$\frac{\partial F_1}{\partial t_1} = b_{1_{\mu_1}} p_1^{(\mu_1)}(t_1; \mathbf{t})\, V_0^1$$

$$+ \sum_{i=2}^{k} \sum_{j=1}^{\mu_i} b_{ij}\, p_1^{(\mu_1)}(t_1; \mathbf{t})\, V_{ij}^1 (-1)^{(\sum_{l=2}^{i-1} \mu_l) + j + 1}.$$

One obtains $\partial F_i / \partial t_i$ similarly for $i=2, 3, ..., k$. Moreover, by construction $p_i^{(\mu_j)}(t_j; \mathbf{t})=0$ for $i \neq j$ and therefore the off-diagonal elements vanish. Thus, in light of Lemma 3, the result is shown.

LEMMA 5. $\mathrm{Deg}(F(\cdot; 0), \Delta_{k,\varepsilon}, 0)=1$.

Proof. For the remainder of the lemma set λ to zero and delete any further reference to λ. We follow Barrow [3] by first showing that the only solution to $F(\mathbf{t})=0$ is $\mathbf{r}$ and then show that $F(\mathbf{t})$ is one-to-one for $\mathbf{t}$ near $\mathbf{r}$.

Then $F_i(\mathbf{t}) = -L_0(p_i(\cdot; \mathbf{t})) = -\sum_{l=1}^{k} p_i(r_l; \mathbf{t})$. By the definition of p_i, $F(\mathbf{r}) = 0$. Suppose that there exists another solution, say $\mathbf{s}$, so that $F(\mathbf{s}) = 0$. By Lemma 1 let $\mathbf{s}$, $\mathbf{a}$ be the corresponding formula (3.1) for L_0.

Then without loss of generality we assume that $r_1 \neq s_1$ and construct $p \in U$ as follows:

$$p[\underbrace{x_1, ..., x_1}_{m_1}, ..., \underbrace{x_i, ..., x_i}_{j}] = 0, \qquad i = 1, ..., s, j = 1, 2, ..., m_i$$

$$p[\mathbf{x}, \underbrace{s_1, ..., s_1}_{\mu_1}, ..., \underbrace{s_i, ..., s_i}_{j}] = 0, \qquad i = 1, 2, ..., k, j = 1, 2, ..., \mu_i$$

$$p(r_i) = 0 \qquad i = 2, 3, ..., k$$

$$p(r_1) \neq 0.$$

Then by (3.1), $L_0(p) = 0$, but $L_0(p) = p(r_1) \neq 0$, a contradiction. Therefore $r_1 = s_1$ and similarly one shows $\mathbf{r} = \mathbf{s}$. We now compute $\partial F/\partial \mathbf{t}$ for $\mathbf{t}$ near $\mathbf{r}$ to show that F is one-to-one near $\mathbf{r}$.

For $\mathbf{t}$ near $\mathbf{r}$, letting $h = r_l - t_l$,

$$\begin{aligned} F_i(\mathbf{t}) &= -\sum_{l=1}^{k} p_i(r_l; \mathbf{t}) \\ &= -\sum_{l=1}^{k} p_i(t_i; \mathbf{t}) - hp_i'(t_l; \mathbf{t}) \\ &\quad + \cdots - \frac{h^{(\mu_l - 1)}}{(\mu_l - 1)} p_i^{(\mu_l - 1)}(t_i; \mathbf{t}) - \frac{hp_i^{(\mu_l)}}{\mu_l!} (t_l; \mathbf{t}) + o(h^{\mu_l}). \end{aligned}$$

By the definition of p_i we have

$$F_i(\mathbf{t}) = -\frac{h^{(\mu_i)}}{\mu_l! \, V_{i_{\mu_i}}} + \sum_{l=1}^{k} o((r_l - t_l)^{\mu_l}).$$

Making the invertible, orientation-preserving change of variables $y_i = (t_i - r_i)^{\mu_i}$ we have

$$F_i(y_i) = \frac{y_i}{\mu_i! \, V_{i_{\mu_i}}} + \sum_{l=1}^{k} o(y_l).$$

Hence

$$\frac{\partial F(o)}{\partial y} = \operatorname{diag}\left(\frac{1}{\mu_1! \, V_{1_{\mu_1}}}, ..., \frac{1}{\mu_k! \, V_{k_{\mu_k}}}\right)$$

and so by the inverse function theorem $F(\mathbf{t})$ is one-to-one for $\mathbf{t}$ near $\mathbf{r}$. As $\deg(F, \Delta_{k,\varepsilon}, o) = \deg(F, \Delta_{k,\varepsilon}, c)$ for c near 0, the degree is one.

Theorem 1 now follows. Lemma 5 and property (iii) of the degree give us $\deg(F(\cdot; 1), \Delta_{k,\varepsilon}, o) = 1$. By properties (i) and (ii) of the degree combined with Lemma 4 we conclude that the equation $F(\mathbf{t}; 1) = 0$ has a unique solution. Thus, by Lemma 1, Theorem 1 follows.

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